Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 57 (2007) 2124–2135
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Relativistic particles and the geometry of 4-D null curves

Angel Ferrández[a], Angel Giménez[b], Pascual Lucas[a,*]

[a] *Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Murcia, Spain*
[b] *Centro de Investigación Operativa, Universidad Miguel Hernández, Avda. Universidad s/n, 03202 Elche, Alicante, Spain*

Received 19 December 2006; received in revised form 19 April 2007; accepted 22 May 2007
Available online 25 May 2007

Abstract

We study actions in $(d+1)$-dimensions associated with null curves, mainly when $d = 3$, whose Lagrangian is a linear function on the curvature of the particle path, showing that null helices are always possible trajectories of the particles. We find Killing vector fields along critical curves of the action which correspond to the linear and the angular momenta of the particle. They provide two constants of the motion which can be interpreted in terms of the mass and the spin of the system. Moreover, we are able to integrate both the Euler–Lagrange equations and the Cartan equations in cylindrical coordinates around a certain plane.

PACS: 04.20.-q; 02.40.-k

Keywords: Lagrangian; Euler–Lagrange equations; Null curve; Critical point; Killing field; Hénon–Heiles systems

1. Introduction

In the past fifteen years, many interesting papers concerning Lagrangians describing spinning particles have been published (see e.g. [1–11] and references therein). In the general situation, as is well known, one has to provide the classical model with the extra bosonic variables. To this end, an interesting hypothesis deals with Lagrangians on higher geometrical invariants to supply those extra degrees of freedom. This approach has the interesting point of view that the spinning degrees of freedom are encoded in the geometry of its world trajectories. The Poincaré and invariance requirements imply that an admissible Lagrangian density F must depend on the extrinsic curvatures of curves in the background gravitational field. In particular, the Lagrangians depending on the first and second curvatures have been intensively studied in the last twenty years. At the beginning those systems were studied as toy models of rigid strings and $(2+1)$-dimensional field theories with the Chern–Simons term, but shortly after, mainly due to the papers by Plyushchay, those systems are of independent interest.

The actions considered before are defined on nonisotropic curves (spacelike or timelike), but on $(d+1)$-spacetimes one can also consider actions defined on null (lightlike) curves. The studies of Lagrangians on these curves begin in the late nineties by considering the simplest geometrical particle model associated with null paths in four-dimensional

* Corresponding author.
E-mail addresses: aferr@um.es (A. Ferrández), a.gimenez@umh.es (A. Giménez), plucas@um.es (P. Lucas).

0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2007.05.006

Minkowski spacetime, [12], where the action is proportional to the pseudo-arc of the particle. The authors obtain the equations of motion and show that they are particular examples of null helices. The same authors consider in [13] this geometrical particle model associated with null curves in $(2+1)$-dimensions. Our main results have been recently exploited to provide a Lagrangian description of the dynamics of geometric models for null curves (see [14]). It is worth pointing out that our latest results form part of a general programme whose seminal paper was [15] (see also [16,17]).

The next step deals with a more complicated three-dimensional system where the action is a linear function on the curvature of the curve. In [18] the authors show that its mass and spin spectra are defined by one-dimensional nonrelativistic mechanics with a cubic potential. Recently, in [19] we obtain, using geometrical methods, a complete description of the relativistic particle paths.

This paper concerns actions in $(d+1)$-dimensions whose Lagrangian is a linear function on the curvature of the particle path. The paper is organized as follows: in Section 2 we present the model, whose action $\mathcal{L} : \Lambda \to \mathbb{R}$ is given by $\mathcal{L}(\gamma) = \int_\gamma (\mu k_1 + \lambda) \mathrm{d}\sigma$, where μ and λ are constants and k_1 stands for the first curvature of the null curve. The equations of motion for these Lagrangians are completely given in $(d+1)$-background gravitational fields. In Section 3 we solve the motion equations and get the null worldlines of the relativistic particles in cylindrical coordinates around a certain hyperplane Π. To this end, we distinguish two cases according to Π is either non-null (space-like or time-like) or null. In Section 4 we give a complete description of critical curves, by using cylindrical coordinates, of the Lagrangian $\mathcal{L}(\gamma)$. Finally, Section 5 is devoted to discussion and concluding remarks.

2. The model and the equations of motion

Let $\mathbb{L}^n$ be an n-dimensional Lorentz–Minkowski space with background gravitational field $\langle , \rangle$ and Levi-Civita connection ∇. First of all, we will describe the geometry of null curves in $\mathbb{L}^n$ in terms of the Cartan frame of the curve (see [20] for details).

Let $\gamma : [a, b] \to \mathbb{L}^n$ be a parametrized null Cartan curve such that the frame $\left\{\gamma'(\sigma), \gamma''(\sigma), \ldots, \gamma^{(n)}(\sigma)\right\}$ is positively oriented, for all $\sigma \in [a, b]$, σ being the pseudo-arc parameter. Let us consider its corresponding Cartan frame $\left\{L = \gamma', W_1, N, W_2, \ldots, W_{n-2}\right\}$, where

$$\langle L, L\rangle = \langle N, N\rangle = 0, \qquad \langle L, N\rangle = -1,$$
$$\langle W_i, L\rangle = \langle W_i, N\rangle = 0, \qquad \left\langle W_i, W_j\right\rangle = \delta_{ij}.$$

The Cartan equations read

$$\begin{aligned} L' &= W_1, \\ W_1' &= -k_1 L + N, \\ N' &= -k_1 W_1 + k_2 W_2, \\ W_2' &= k_2 L + k_3 W_3, \\ W_i' &= -k_i W_{i-1} + k_{i+1} W_{i+1} \quad i \in \{3, \ldots, n-3\}, \\ W_{n-2}' &= -k_{n-2} W_{n-3}, \end{aligned} \tag{1}$$

where $()'$ means covariant derivative and k_i are the *Cartan curvatures* of the curve. The fundamental theorem for null curves tells us that $\{k_1, \ldots, k_{n-2}\}$ determines completely the null curve up to Lorentzian transformations (see [20]). Even more, given functions $\{k_1, \ldots, k_{n-2}\}$, we can always construct a null curve, parametrized by the pseudo-arc length parameter σ, whose curvatures functions are precisely $\{k_1, \ldots, k_{n-2}\}$ (see [20, Theorem 3]). Then any local geometrical scalar defined along null curves can always be expressed as a function of its curvatures and derivatives.

In this section we analyse mechanical systems with Lagrangians depending linearly on the first curvature k_1 of the null curve. The space of elementary fields in this model is the set Λ of all null Cartan curves satisfying given first order boundary data to drop out the boundary terms which appear when computing the first order variation of the action.

For the sake of simplicity γ will also denote a variation of γ by null curves $\gamma = \gamma(s, \omega) : [a, b] \times (-\varepsilon, \varepsilon) \to \mathbb{L}^n$, where $\gamma(s, 0)$ is the reparametrization of $\gamma(\sigma)$. Associated with such a variation is the variational vector field $V(s) = V(s, 0)$, where $V = V(s, \omega) = \frac{\partial \gamma}{\partial \omega}(s, \omega)$. Let η be the differentiable function satisfying $\frac{\partial \gamma}{\partial s}(s, \omega) = \eta(s, \omega) L(s, \omega)$. Then we will write down $\gamma(\sigma, \omega)$, $k(\sigma, \omega)$, $V(\sigma, \omega)$, etc., for the corresponding pseudo-arc length parameter.

The actions $\mathcal{L}$ for the curve depend locally on its geometry and they possess various symmetries, both local and global. The local symmetry is reparametrization invariance and it restricts severely the form of $\mathcal{L}$. We consider the action $\mathcal{L} : \Lambda \to \mathbb{R}$ given by

$$\mathcal{L}(\gamma) = \int_\gamma (\mu k_1 + \lambda)\mathrm{d}\sigma, \tag{2}$$

μ and λ both being constant. The simplest action describing the motion of a particle is achieved when it is proportional to the pseudo-arc length parameter (i.e. $\mu = 0$), which has been studied by Nersessian and Ramos in [12,13] when $n = 2, 3$. When the action is linear on the curvature of the particle path, some advances have been achieved in [18, 19].

A null curve γ is said to be a critical point of the action $\mathcal{L}$ when

$$\left.\frac{\mathrm{d}}{\mathrm{d}\omega}\right|_{\omega=0} \mathcal{L}(\gamma_\omega) = \left.\frac{\mathrm{d}}{\mathrm{d}\omega}\right|_{\omega=0} \int_{\gamma_\omega} (\mu k_1 + \lambda)\mathrm{d}\sigma = 0,$$

for all variation throughout null curves γ_ω of γ.

Now we present a useful technical result.

Lemma 1. *Using the above notation, the following assertions hold:*

(a) $0 = \langle \nabla_L V, L \rangle$;

(b) $\dfrac{\partial \eta}{\partial \omega} = V(\eta) = -\dfrac{1}{2} h\eta, \qquad h = -\left\langle \nabla_L^2 V, W_1 \right\rangle$;

(c) $\dfrac{\partial k_1}{\partial \omega} = \left\langle \nabla_L^3 V, N \right\rangle + k_1 \langle \nabla_L V, N \rangle + k_1 h - \dfrac{1}{2} L(L(h))$;

(d) $\dfrac{\partial k_2}{\partial \omega} = \left\langle \nabla_L^4 V, W_2 \right\rangle + 2k_1 \left\langle \nabla_L^2 V, W_2 \right\rangle + k_1' \langle \nabla_L V, W_2 \rangle + 2k_2 h.$

A variational vector field V along γ which infinitesimally preserves the causal character, the pseudo-arc length parameter and the curvatures of γ is said to be a Killing vector field along γ. Hence Killing vector fields along γ are characterized by the following equations:

$$\langle \nabla_L V, L \rangle = 0,$$
$$V(\eta) = 0,$$
$$V(k_i) = 0, \quad i = 1, \ldots, n-2.$$

As we will see, Killing vector fields will play an important role to integrate the Euler–Lagrange and Cartan equations. In particular, we are specially interested in the four-dimensional situation.

Proposition 2. *Let γ be an immersed null curve in $\mathbb{L}^4$. A vector field V is Killing along γ if and only if it extends to a Killing vector field $\tilde{V}$ on $\mathbb{L}^4$.*

The same conclusion holds by considering a complete, simply connected, Lorentzian space form.

To compute the first-order variation of this action along the elementary fields space Λ, and so the field equations describing the dynamics of the particle, we use a standard argument involving some integrations by parts. Then the Cartan equations yield

$$\mathcal{L}'(0) = \frac{1}{2}[\Omega(\gamma, V)]_a^b - \frac{1}{2} \int_a^b \langle V, \mathcal{E}_1(\gamma)L + \mathcal{E}_2(\gamma)W_2 + \mathcal{E}_3(\gamma)W_3 + \mathcal{E}_4(\gamma)W_4 \rangle \,\mathrm{d}\sigma, \tag{3}$$

where

$$\begin{aligned} \mathcal{E}_1(\gamma) &= \mu k_1''' + 2\mu k_2 k_2' + 3\mu k_1 k_1' - \lambda k_1', \\ \mathcal{E}_2(\gamma) &= 2\mu k_2'' - k_2(2\mu k_3 - \mu k_1 + \lambda), \\ \mathcal{E}_3(\gamma) &= 2\mu(k_2 k_3' - 2k_2' k_3), \\ \mathcal{E}_4(\gamma) &= k_2 k_3 k_4, \end{aligned} \tag{4}$$

and the boundary term reads

$$\Omega(\gamma, V) = \left\langle \nabla_L^3 V, \mu W_1 \right\rangle + \left\langle \nabla_L^2 V, -\mu k_1 L + 3\mu N \right\rangle + \langle \nabla_L V, (\mu k_1 + \lambda) W_1 - \mu k_2 W_2 \rangle + \langle V, P_1 \rangle, \tag{5}$$

where P_1 is the vector field given by

$$P_1 = \left(\mu k_1'' + \mu k_1^2 - \lambda k_1 \right) L - \mu k_1' W_1 + (\mu k_1 - \lambda) N + 2\mu k_2' W_2 + 2\mu k_2 k_3 W_3, \tag{6}$$

and V stands for a generic variational vector field along γ.

To drop $[\Omega(\gamma, V)]_a^b$ we have to consider curves with the same endpoints and having the same Cartan frame there. Under these conditions, the first-order variation reads

$$\mathcal{L}'(0) = -\frac{1}{2} \int_a^b \langle V, \mathcal{E}_1(\gamma) L + \mathcal{E}_2(\gamma) W_2 + \mathcal{E}_3(\gamma) W_3 + \mathcal{E}_4(\gamma) W_4 \rangle \, d\sigma.$$

As a consequence we have

Theorem 3. *A null curve $\gamma \in \Lambda$ is critical for the linear action $\mathcal{L}(\gamma)$ in $\mathbb{L}^n$ if and only if the following statements hold:*

(i) *W_i, N and k_j are well defined along the whole trajectory;*
(ii) *The following differential equations are fulfilled:*

$$\mathcal{E}_1(\gamma) = 0, \qquad \mathcal{E}_2(\gamma) = 0, \qquad \mathcal{E}_3(\gamma) = 0, \qquad \mathcal{E}_4(\gamma) = 0.$$

These equations are called the *Euler–Lagrange equations*. The following is an easy consequence from the last equation of (4):

Corollary 4. *The critical points for the linear action $\mathcal{L}(\gamma)$ in $\mathbb{L}^n$ lie in a Lorentzian subspace of dimension not greater than five.*

By considering the special case where the action is constant ($\mu = 0$), the Euler–Lagrange equations are reduced to

$$-\lambda k_1' = 0, \qquad -\lambda k_2 = 0, \qquad k_2 k_3 k_4 = 0.$$

As a consequence we have

Theorem 5. *The critical points for the constant action in $\mathbb{L}^n$ are just null helices in* 3*-dimensional Lorentzian linear subspaces.*

We have made a more general treatment of Lagrangian in the 3-dimensional case (see [21]). There we have explicitly obtained all solutions for a linear action as well as got remarkable progress regarding other more difficult Lagrangians. Therefore, it seems reasonable to investigate the critical points of the linear action in the 4-dimensional case.

3. Linear action in $\mathbb{L}^4$

By reconsidering the action (2) in $\mathbb{L}^4$ we get $k_3 = 0$ (and $k_4 = 0$). Without loss of generality we normalize the constant μ by one and replace k_1 and k_2 by k and τ, respectively. Then the action (2) rewrite as

$$\mathcal{L}(\gamma) = \int_\gamma (k + \lambda) d\sigma, \tag{7}$$

and the Euler–Lagrange equations as

$$k''' + 2\tau\tau' + (3k - \lambda)k' = 0, \qquad 2\tau'' + (k - \lambda)\tau = 0. \tag{8}$$

Since the first equation can be easily integrated, the Euler–Lagrange equations state as

$$k'' + \tau^2 + \frac{3}{2}k^2 - \lambda k + c = 0, \qquad 2\tau'' + (k - \lambda)\tau = 0, \tag{9}$$

where c is an integration constant.

Now we are searching for nontrivial ($\tau \neq 0$) solutions. We first find easy solutions when $k = \lambda$. Then τ is a nonzero constant and we get Cartan helices in $\mathbb{L}^4$. From now on, we will assume that $k \neq \lambda$. A straightforward computation shows that $P_1 = (k'' + k^2 - \lambda k)L - k' W_1 + (k - \lambda)N + 2\tau' W_2$ satisfies that $\nabla_L P_1 = \mathcal{E}_1(\gamma)L + \mathcal{E}_2(\gamma)W_2$. Therefore, P_1 is a constant Killing vector field if and only if γ is a solution of the Euler–Lagrange equations. We can apply the Noether argument relating rotational symmetries of $\mathcal{L}$ to constant of motions along γ. The variational vector field associated to variations generated by a one-parameter family of rotations is given by $\gamma \wedge Z_1 \wedge Z_2$, where Z_1 and Z_2 are constant vector fields. As γ satisfies the Euler–Lagrange equations and the action is rotationally invariant, we find

$$\Omega(\gamma, \gamma \wedge Z_1 \wedge Z_2) = \langle (k + \lambda)\, L \wedge W_1 \wedge Z_1 - 2\tau\, L \wedge W_2 \wedge Z_1 + 2W_1 \wedge N \wedge Z_1 + \gamma \wedge P_1 \wedge Z_1, Z_2 \rangle$$

is constant for any constant vector field Z_2. Then the new vector field

$$Y = (k + \lambda)\, L \wedge W_1 \wedge Z_1 - 2\tau\, L \wedge W_2 \wedge Z_1 + 2\, W_1 \wedge N \wedge Z_1 + \gamma \wedge P_1 \wedge Z_1 \tag{10}$$

is also constant along γ for any constant vector field Z_1. In particular, we can replace Z_1 by P_1 to get the constant Killing vector field

$$P_2 = \left(2(k + \lambda)\tau' - 2\tau k'\right) L + 2(k - \lambda)\tau W_1 + 4\tau' N + \left(2k'' + 3k^2 - 2\lambda k - \lambda^2\right) W_2.$$

Observe that P_1 is orthogonal to P_2 unless both are null. This exceptional case will be considered later, where we will find explicit solutions. Then we will assume that one of them is non-null. By replacing Z_1 by P_2 into (10) we obtain a new constant vector field

$$\begin{aligned} X = &\left(2(k + \lambda)k'' + 4(k - \lambda)\tau^2 + 3k^3 + \lambda k^2 - 3\lambda^2 k - \lambda^3\right) L + 8\tau\tau' W_1 \\ &+ \left(4k'' + 6k^2 - 4\lambda k - 2\lambda^2\right) N + \left(8(k + \lambda)\tau' - 4\tau k'\right) W_2 + P_1 \wedge P_2 \wedge \gamma. \end{aligned}$$

Set $J = X - P_1 \wedge P_2 \wedge \gamma$. As J is the sum of a translational and a rotational vector fields, then J is a Killing vector field. It is easy to check that $\langle P_1, P_2 \rangle = \langle P_2, J \rangle = 0$ and $\langle P_2, P_2 \rangle = -\langle P_1, J \rangle$. We write down

$$\langle P_1, P_1 \rangle = \varepsilon_1 p_1^2, \qquad \langle P_1, J \rangle = \omega, \qquad \langle P_2, P_2 \rangle = \varepsilon_2 p_2^2,$$

where $\varepsilon_1, \varepsilon_2 \in \{-1, 0, 1\}$, $p_1 = \|P_1\|$, $p_2 = \|P_2\|$ and $\omega = -\varepsilon_2 p_2^2$. The constants p_1 and ω represent the constants of motion of the relativistic particle, which correspond to the mass and spin of the system.

Proposition 6. *Let γ be a null curve in $\mathbb{L}^4$ with curvatures k and τ. Assume that $\{L, W_1, P_1, J\}$ is a set of linearly independent vector fields. Then γ is a solution of the Euler–Lagrange equations if and only if $\langle P_1, P_1 \rangle$ and $\langle P_1, J \rangle$ are both constant.*

Proof. We first observe that γ is a solution of the Euler–Lagrange equations if and only if the vector field P_1 along γ satisfies $\nabla_L P_1 = 0$. To see that holds we have to show that $\langle \nabla_L P_1, P_1 \rangle = \langle \nabla_L P_1, J \rangle = 0$, because $\{L, W_1, P_1, J\}$ is a linearly independent system. A straightforward computation gives $\langle \nabla_L P_1, J \rangle = \langle P_1, \nabla_L J \rangle$, which leads to

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} \langle P_1, P_1 \rangle = 2 \langle \nabla_L P_1, P_1 \rangle, \qquad \frac{\mathrm{d}}{\mathrm{d}\sigma} \langle P_1, J \rangle = 2 \langle \nabla_L P_1, J \rangle,$$

and the result follows. □

Therefore, asking for $\langle P_1 \wedge J \wedge L, W_1 \rangle \neq 0$, or said otherwise

$$4(\lambda - k)\tau k' - 4\tau'(2k'' + k^2 - 2\lambda k + \lambda^2) \neq 0,$$

the Euler–Lagrange equations are equivalent to those given by the constants of motion

$$\begin{aligned} &4(\tau')^2 + (k')^2 - 2(k - \lambda)(k'' + k^2 - \lambda k) - \varepsilon_1 p_1^2 = 0, \\ &16\left((k + \lambda)\tau' - \tau k'\right)\tau' - \left(2k'' + 3k^2 - 2\lambda k - \lambda^2\right)^2 - 4(k - \lambda)^2\tau^2 - \omega = 0. \end{aligned} \tag{11}$$

Combining (9) with (11) we get the equations of the motion

$$\begin{aligned} &4(\tau')^2 + (k')^2 + (k-\lambda)(k^2 + 2(\tau^2 + c)) - \varepsilon_1 p_1^2 = 0, \\ &16\left((k+\lambda)\tau' - \tau k'\right)\tau' - \left(2\tau^2 + 2c + \lambda^2\right)^2 - 4(k-\lambda)^2\tau^2 - \omega = 0. \end{aligned} \tag{12}$$

To study this system we adopt a Hamiltonian point of view, because it reminds us a Hénon–Heiles system. To this end, we will provisionally introduce a more conventional notation. First, we set

$$q_1 = k, \qquad q_2 = 2\tau, \qquad p_1 = k', \qquad p_2 = 2\tau'. \tag{13}$$

Then, the Euler–Lagrange equations (9) take the form

$$q_1'' + \frac{1}{4}q_2^2 + \frac{3}{2}q_1^2 - \lambda q_1 + c = 0, \qquad q_2'' + \frac{1}{2}q_2(q_1 - \lambda) = 0. \tag{14}$$

The Hamiltonian function of the system is going to be the first constant of motion

$$H(q,p) = \frac{1}{2}\langle P_1, P_1\rangle = \frac{1}{2}\left(p_1^2 + p_2^2 - \lambda q_1^2 - \frac{\lambda}{2}q_2^2\right) + \frac{1}{4}q_1 q_2^2 + \frac{1}{2}q_1^3 + c(q_1 - \lambda). \tag{15}$$

According to Eq. (14) we obtain

$$\begin{aligned} p_1' &= q_1'' = -\frac{1}{4}q_2^2 - \frac{3}{2}q_1^2 + \lambda q_1 - c = -\frac{\partial H}{\partial q_1}, \\ p_2' &= q_2'' = -\frac{1}{2}q_2(q_1 - \lambda) = -\frac{\partial H}{\partial q_2}. \end{aligned}$$

Thus, we conclude that the dynamics of this system obeys a classical two freedom degree Hamiltonian, the sum of kinetic and potential energies, where the potential is a cubic polynomial in the position variables q_1, q_2, that is,

$$\begin{aligned} H &= \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2), \\ V(q_1, q_2) &= \frac{1}{2}\left(-\lambda q_1^2 - \frac{\lambda}{2}q_2^2\right) + \frac{1}{4}q_1 q_2^2 + \frac{1}{2}q_1^3 + c(q_1 - \lambda). \end{aligned}$$

Observe that when $c = 0$ this system corresponds to a particular case of the general Hénon–Heiles Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + Aq_1^2 + Bq_2^2) + Dq_1 q_2^2 - \frac{C}{3}q_1^3,$$

where A, B, C and D are constant coefficients. The integrability of the system depends on the values of the coefficients. The equations of motion are integrable in three cases: (i) $A = B$ and $D = -C$; (ii) $6D = -C$; and (iii)$16D = -C$ and $6A = B$. Note that Hamiltonian (15), when $c = 0$, fit in case (ii), since $D = 1/4$ and $C = -3/2$. By integrability here we mean existence of a second (global) integral of motion and, in this case, the Liouville theorem implies that the problem can be solved by quadratures. For a recent treatment of more general integrable cases we refer the reader to [22].

Going back to the primitive notation, we proceed to drop the curvature τ by taking derivative twice in the first equation of (9). Then, from (9) and the first equation in (12) we obtain a fourth-order ODE in k

$$k^{(4)} + (5k - 3\lambda)k'' + \frac{5}{2}(k')^2 + \frac{5}{2}k^3 - \frac{9}{2}\lambda k^2 + (2\lambda^2 + c)k + \varepsilon_1 p_1^2 - c\lambda = 0, \tag{16}$$

so that $k^{(4)}$ is a polynomial in k, k' and k''.

Observe that we can identify this Hamiltonian system with solitons (i.e. travelling wave solutions of the form $u(x,t) = u(x - ct)$) of the fifth-order evolution conservative partial differential equation

$$u_t + \left(u_{xxxx} + (5u - 3\lambda)u_{xx} + \frac{5}{2}u_x^2 + \frac{5}{2}u^3 - \frac{9}{2}\lambda u^2 + (2\lambda^2 + c)u\right)_x = 0,$$

which is a soliton equation called fifth-order KdV equation (KdV5). The general solution of (16) has been obtained in terms of hyperelliptical functions (see [23]) by separation of the variables of the Hamilton–Jacobi equation in parabolic coordinates.

We do not pretend to find all solutions of this system; however, we will give an explicit solution of a quite interesting particular case when P_1 and P_2 are null. In this case, P_1 and P_2 should be collinear and so there does not exist a rotational plane. We assume that $P_2 = \beta P_1$, where β is a constant. Equalling coefficients of these vector fields we get

$$\begin{aligned} &\text{(a) } 2(k+\lambda)\tau' - 2\tau k' = \beta\left(k'' + k^2 - \lambda k\right),\\ &\text{(b) } 2\tau(k-\lambda) = -\beta k',\\ &\text{(c) } 4\tau' = \beta(k-\lambda),\\ &\text{(d) } 2k'' + 3k^2 - 2\lambda k - \lambda^2 = 2\beta\tau'. \end{aligned} \tag{17}$$

Taking derivative in (17)(c) and bringing it to the second equation of (9) we deduce that

$$\tau = -\frac{\beta}{2}\frac{k'}{k-\lambda}. \tag{18}$$

Take derivative here and multiply both terms by $k'/(k-\lambda)$ to obtain

$$k' = -2\frac{k'}{k-\lambda}\partial_t\left(\frac{k'}{k-\lambda}\right).$$

We may now compute an integral of the latter equation

$$(k')^2 = -(k-\lambda)^2(k-d), \tag{19}$$

d being a constant. We take again derivative to get

$$k'' = -(k-\lambda)(k-d) - \frac{1}{2}(k-\lambda)^2. \tag{20}$$

Finally, combining (18)–(20) it yields (17)(a) and (17)(b). The fourth equation of (17) holds if and only if $d = \frac{\beta^2}{4} - \lambda$ holds. Now, from (17)(b) and (19), it is easy to check that

$$\tau^2 = -\frac{\beta^2}{4}(k-d) = -\frac{\beta^2}{4}\left(k + \lambda - \frac{\beta^2}{4}\right). \tag{21}$$

Furthermore, the constant c appearing in (9) is related to β and d by

$$c = \frac{\lambda^2}{2} - d^2 = \frac{\lambda^2}{2} - \left(\frac{\beta^2}{4} - \lambda\right)^2.$$

Examining Eqs. (19) and (21) we can find explicit solutions depending on both λ and β.

Case $d = \lambda$.

This means that $\beta^2/8 = \lambda$ and the solutions are

$$k(\sigma) = \lambda - \frac{4}{(\sigma+e)^2}, \qquad \tau(\sigma)^2 = \frac{8\lambda}{(\sigma+e)^2} = \frac{\beta^2}{(\sigma+e)^2}.$$

Case $d > \lambda$.

Now $\beta^2/8 > \lambda$ and there are two solutions depending on $k(\sigma) \in (-\infty, \lambda)$ or $k(\sigma) \in (\lambda, d]$

$$k(\sigma) = \left(\frac{\beta^2}{4} - \lambda\right) - \left(\frac{\beta^2}{4} - 2\lambda\right)\coth^2\left(\frac{1}{4}\sqrt{\beta^2 - 8\lambda}\,(\sigma + e)\right),$$
$$\tau(\sigma)^2 = \frac{\beta^2}{4}\left(\frac{\beta^2}{4} - 2\lambda\right)\coth^2\left(\frac{1}{4}\sqrt{\beta^2 - 8\lambda}\,(\sigma + e)\right),$$
$$k(\sigma) = \left(\frac{\beta^2}{4} - \lambda\right) - \left(\frac{\beta^2}{4} - 2\lambda\right)\tanh^2\left(\frac{1}{4}\sqrt{\beta^2 - 8\lambda}\,(\sigma + e)\right),$$
$$\tau(\sigma)^2 = \frac{\beta^2}{4}\left(\frac{\beta^2}{4} - 2\lambda\right)\tanh^2\left(\frac{1}{4}\sqrt{\beta^2 - 8\lambda}\,(\sigma + e)\right),$$

respectively.

Case $d < \lambda$.

Then $\beta^2/8 < \lambda$ and the solutions are

$$k(\sigma) = \left(\frac{\beta^2}{4} - \lambda\right) - \left(2\lambda - \frac{\beta^2}{4}\right)\tan^2\left(\frac{1}{4}\sqrt{8\lambda - \beta^2}\,(\sigma + e)\right),$$
$$\tau(\sigma)^2 = \frac{\beta^2}{4}\left(2\lambda - \frac{\beta^2}{4}\right)\tan^2\left(\frac{1}{4}\sqrt{8\lambda - \beta^2}\,(\sigma + e)\right).$$

It is worth investigating how the above particular solutions can be seen as solutions of (16). Set $\varphi = k'' + \frac{3}{2}k^2 - 3\lambda k + 2\lambda^2 + c$ and consider the differential operator

$$\mathcal{D} = \partial_{\sigma\sigma\sigma} + 2k\partial_\sigma + k' I.$$

Note that $\mathcal{D}$ is the Jacobi operator of the second Poisson structure of the KdV equation. It is easy to see that

$$\begin{aligned}\mathcal{D}(\varphi) &= \varphi''' + 2k\varphi' + k'\varphi \\ &= k^{(5)} + (5k - 3\lambda)k''' + 10k'k'' + \left(\frac{15}{2}k^2 - 9\lambda + 2\lambda^2 + c\right)k' \\ &= \partial_\sigma\left(k^{(4)} + (5k - 3\lambda)k'' + \frac{5}{2}(k')^2 + \frac{5}{2}k^3 - \frac{9}{2}\lambda k^2 + (2\lambda^2 + c)k + \varepsilon_1 p_1^2 - c\lambda\right).\end{aligned}$$

It follows that k is a solution of (16) if and only if $\mathcal{D}(\varphi) = 0$. Observe that (20) reads

$$\psi = k'' + \frac{3}{2}k^2 - (2\lambda + d)k + \lambda d + \frac{\lambda^2}{2} = 0.$$

Then we have $\varphi = \psi + \phi$, where $\phi = (d - \lambda)(k - (d + 2\lambda))$, and $\mathcal{D}(\phi) = (d - \lambda)\psi'$. We conclude that $\mathcal{D}(\varphi) = \mathcal{D}(\phi) = 0$ provided $\psi = 0$.

4. Solving the natural equations

In order to find the critical null curves of the Lagrangian (7) we have to introduce cylindrical coordinates around the rotational plane $\Pi = \text{span}\{P_1, P_2\}$ spanned by P_1 and P_2. Such $\Pi = \text{span}\{P_1, P_2\}$ can be spacelike, timelike or null depending on the causal character of P_1 and P_2. Bearing in mind that $\langle P_1, P_2\rangle = 0$, in the following table we collect all possibilities.

		P_1		
		spacelike	timelike	null
P_2	spacelike	Π is spacelike	Π is timelike	Π is null
	timelike	Π is timelike	never	never
	null	Π is null	never	never

Proposition 7. *There exist a suitable translation of the coordinates origin for which the Killing vector field J can be expressed as follows:*

$$J = -P_1 \wedge P_2 \wedge \gamma + \varepsilon_1 \omega P_1^* = -P_1 \wedge P_2 \wedge \gamma - \varepsilon_1 \varepsilon_2 p_2^2 P_1^*, \tag{22}$$

where P_1^* have the same causal character as P_1 and satisfies that $\langle P_1, P_1^* \rangle = \varepsilon_1$.

Proof. Let $\tilde{\gamma} = \gamma - Y$ be a general translation, Y being a constant vector field. Then we have

$$J = -P_1 \wedge P_2 \wedge \gamma + X = -P_1 \wedge P_2 \wedge (\tilde{\gamma} + Y) + X = -P_1 \wedge P_2 \wedge \tilde{\gamma} - P_1 \wedge P_2 \wedge Y + X.$$

We are going to find vector fields P_1^* and Y satisfying that $X = \varepsilon_1 \omega P_1^* + P_1 \wedge P_2 \wedge Y$. To do that we distinguish two cases.

P_1 is not null. The vector fields P_1 and X are in $P_2^{\perp}$. As P_1 is not null, we obtain the splitting $P_2^{\perp} = \text{span}\{P_1\} \perp P_1^{\perp}$, where $P_1^{\perp}$ stands for the orthogonal space to P_1 in $P_2^{\perp}$. Therefore, we can find a vector field Y and a constant μ such that

$$X = P_1 \wedge P_2 \wedge Y + \mu P_1.$$

We finish by taking $P_1^* = \frac{\varepsilon_1 \mu}{\omega} P_1$.

P_1 is null. Now there exists P_1^* (not unique) such that $\langle P_1, P_1^* \rangle = \varepsilon_1 = \pm 1$ and $P_2^{\perp} = P_1^{\perp} \oplus \text{span}\{P_1^*\}$. As $X \in P_2^{\perp}$, then there exists a vector field Y and a constant μ so that

$$X = P_1 \wedge P_2 \wedge Y + \mu P_1^*.$$

It follows that $\mu = \varepsilon_1 \omega$, because $\langle X, P_1 \rangle = \langle J, P_1 \rangle = \omega$. □

Next step is devoted to find suitable expressions in cylindrical coordinates of the critical points of $\mathcal{L}$. The preceding result will be extremely important. We will consider two cases depending on the causal character of the rotational plane Π.

4.1. Π is not degenerate

Let (r, θ, z, y) be the coordinates in $\mathbb{L}^4$ given by

$$\begin{aligned} X(r, \theta, z, y) &= (r \cosh\theta, r \sinh\theta, z, y), \quad r \neq 0, \\ X(r, \theta, z, y) &= (z, r \cos\theta, r \sin\theta, y), \quad r > 0, \text{ and } \theta \in (0, 2\pi). \end{aligned} \tag{23}$$

These will be called the *non-degenerate cylindrical coordinates* around the plane spanned by $\{\partial_y, \partial_z\}$. We can assume that P_1 and P_2 are collinear with ∂_z and ∂_y, respectively, interchanging z and y if need be. Then, it is no difficult to see that

$$\langle \partial_\theta, \partial_\theta \rangle = -\varepsilon_1 \varepsilon_2 r^2, \qquad \langle \partial_r, \partial_r \rangle = 1, \qquad \langle \partial_z, \partial_z \rangle = \varepsilon_1, \qquad \langle \partial_y, \partial_y \rangle = \varepsilon_2,$$

vanishing all remaining metric products. Hence, we can write

$$P_1 = p_1 \partial_z, \qquad P_2 = p_2 \partial_y, \qquad J = -p_2 \left(p_1 \partial_\theta + \frac{\varepsilon_1 \varepsilon_2 p_2}{p_1} \partial_z \right),$$

which along with $L = r' \partial_r + \theta' \partial_\theta + z' \partial_z + y' \partial_y$ leads to

$$\begin{aligned} &\langle J, J \rangle = \varepsilon_1 j_2^2 \left(\frac{p_2^2}{p_1^2} - \varepsilon_2 p_1^2 r^2 \right), \qquad \langle L, P_1 \rangle = \varepsilon_1 p_1 z', \\ &\langle L, P_2 \rangle = \varepsilon_2 p_2 y', \qquad \langle L, J \rangle = \varepsilon_2 p_2 \left(\varepsilon_1 p_1 r^2 \theta' - \frac{p_2}{p_1} z' \right). \end{aligned} \tag{24}$$

All these equations yield the following result:

Theorem 8. *Let* $\gamma : I \longrightarrow \mathbb{L}^4$ *be a critical point of* $\mathcal{L}$ *and* Π *a non-null plane. Then* γ *can be described in cylindrical coordinates around* Π *as follows:*

$$r^2 = \frac{\varepsilon_2}{p_1^2}\left(\frac{p_2^2}{p_1^2} - \frac{\varepsilon_1 \langle J, J\rangle}{p_2^2}\right), \qquad z' = \frac{\varepsilon_1 \langle L, P_1\rangle}{p_1},$$
$$y' = \frac{\varepsilon_2 \langle L, P_2\rangle}{p_2}, \qquad \theta' = \frac{1}{p_1 p_2 r^2}\left(\frac{p_2^2 \langle L, P_1\rangle}{p_1^2} + \varepsilon_1\varepsilon_2 \langle L, J\rangle\right), \tag{25}$$

where

$$\langle J, J\rangle = 64\tau^2(\tau')^2 + 16\left(\tau k' - 2(k+\lambda)\tau'\right)^2 + 4\left(2\tau^2 + \lambda^2 + 2c\right)\left((k+\lambda)(2\tau^2 - \lambda^2 - 2c) - 8\lambda\tau^2\right),$$
$$\langle L, P_1\rangle = -(k-\lambda), \qquad \langle L, P_2\rangle = -4\tau', \qquad \langle L, J\rangle = 2\left(2\tau^2 + \lambda^2 + 2c\right). \tag{26}$$

4.1.1. Π is degenerate

We will consider the coordinates (r, θ, z, y) given by

$$X(r, \theta, z, y) = \left(z - \frac{\varepsilon r}{2}(\theta^2 + 1), z - \frac{\varepsilon r}{2}(\theta^2 - 1), -\varepsilon r\theta, y\right), \quad \text{where } r \in \mathbb{R} \setminus \{0\} \text{ and } \theta, z, y \in \mathbb{R}, \tag{27}$$

satisfying

$$\langle \partial_\theta, \partial_\theta\rangle = r^2, \qquad \langle \partial_y, \partial_y\rangle = 1, \qquad \langle \partial_r, \partial_z\rangle = \varepsilon_1,$$

vanishing all remaining metric products. As above, we distinguish two subcases depending on the causal character of P_1.

P_1 *is spacelike*. We may assume, without loss of generality, that P_1 and ∂_y are collinear as well as P_2 and ∂_z. Then we can write

$$P_1 = p_1\partial_y, \qquad P_2 = b\partial_z, \qquad J = -bp_1\partial_\theta.$$

Bearing in mind that $L = r'\partial_r + \theta'\partial_\theta + z'\partial_z + y'\partial_y$, we deduce

$$\langle J, J\rangle = b^2 p_1^2 r^2, \qquad \langle L, P_1\rangle = p_1 y',$$
$$\langle L, P_2\rangle = \varepsilon_1 b r', \qquad \langle L, J\rangle = -bp_1 r^2\theta'. \tag{28}$$

P_1 *is null.* As above, we may assume that P_1 and P_2 are collinear with ∂_z and ∂_y, respectively, and choose $P_1^* = (-\frac{\varepsilon_1}{a}, 0, -\frac{\varepsilon_1}{a}, 0)$. Hence, $P_1 = a\partial_z$, $P_2 = p_2\partial_y$ and we have

$$P_1^* = -\frac{\varepsilon_1}{2a}(\theta - 1)^2\partial_z - \frac{\theta - 1}{ar}\partial_\theta + \frac{1}{a}\partial_r,$$
$$J = \frac{p_2^2}{2a}(\theta - 1)^2\partial_z + \left(ap_2 + \frac{\varepsilon_1 p_2^2}{ar}(\theta - 1)\right)\partial_\theta - \frac{\varepsilon_1 p_2^2}{a}\partial_r.$$

An easy computation shows that

$$\langle J, J\rangle = -p_2^2 r\left(a^2 r + 2\varepsilon_1 p_2(\theta - 1)\right),$$
$$2a\langle L, J\rangle = -2p_2^2 z' + 2r(a^2 r p_2 + \varepsilon_1 p_2^2(\theta - 1))\theta' + \varepsilon_1 p_2^2(\theta - 1)^2 r',$$
$$\langle L, P_1\rangle = \varepsilon_1 a r'$$
$$\langle L, P_2\rangle = p_2 y',$$
$$\langle L, L\rangle = r^2(\theta')^2 + 2\varepsilon_1 r' z' + (y')^2. \tag{29}$$

This yields the following result:

Theorem 9. *Let $\gamma : I \longrightarrow \mathbb{L}^4$ be a critical point of $\mathcal{L}$ and Π a degenerate plane. Then γ can be described in cylindrical coordinates around Π as follows:*

$p_1 = 0,\ p_2 \neq 0$	$p_1 \neq 0,\ p_2 = 0$
$r' = \frac{\varepsilon_1}{a}\langle L, P_1\rangle$	$r^2 = \frac{\langle J, J\rangle}{b^2 p_1^2}$
$y' = \frac{\langle L, P_2\rangle}{p_2}$	$y' = \frac{\langle L, P_1\rangle}{p_1}$
$\theta = 1 - \frac{\varepsilon_1}{2p_2 r}\left(\frac{\langle J, J\rangle}{p_2^2} + a^2 r^2\right)$	$\theta' = -bp_1 \frac{\langle L, J\rangle}{\langle J, J\rangle}$
$z' = -\frac{2}{a\langle L, P_1\rangle}\left(r^2(\theta')^2 + \frac{\langle L, P_2\rangle^2}{p_2^2}\right)$	$z' = -\frac{b}{2\langle L, P_2\rangle}\left(\frac{\langle L, P_1\rangle^2}{p_1^2} + \frac{\langle L, J\rangle^2}{\langle J, J\rangle}\right)$

(30)

where $\langle J, J\rangle$, $\langle L, P_1\rangle$, $\langle L, P_2\rangle$ and $\langle L, J\rangle$ are given by Eqs. (28) and (29).

We conclude by declaring that if we were able to integrate the differential equations satisfied by curvatures, then we should find the curve in cylindrical coordinates by quadratures.

5. Discussion and outlook

We have studied actions in $(d + 1)$-dimensions whose Lagrangians are linear functions on the curvature of the particle path, completing previous works [12,18,19]. We have shown that null helices [20,24] are always possible trajectories of the particles. Otherwise, the non-zero vector field P_1, obtained from the Euler–Lagrange equation, possesses a nonvanishing space-like component orthogonal to the light-like particle trajectory, which seems to be a manifestation of a generic feature of higher-derivative theories. This vector field can be interpreted as the linear momentum of the particle since it is constant along the curve, which agrees with the conserved linear momentum law. Then, the constants of motion turn out to be the mass and spin of the particle. We have obtained massive, massless and tachyonic states, which correspond to space-like, null and time-like momentum vector P_1, respectively. Similar results were shown by Plyushchay in [7] for time-like trajectories.

In the simplest geometrical particle model (the constant case) we show that the worldline of the particle is a 3-dimensional Cartan helix whose axis is spanned by the vector P_1. This was already shown by Nersessian and Ramos using a Hamiltonian formulation, but here we offer an alternative proof which exploits the geometry of the particle path. In the $(d + 1)$-dimensional linear case we give the Euler–Lagrange equations (motion equations) of the system. Furthermore, when $d = 3$ we describe the nature of the motion equations as a two-dimensional nonrelativistic mechanical system with a cubic potential, showing that is completely integrable in the Liouville sense and solving the Euler–Lagrange equations in a particular case. We also integrate the Cartan equations in cylindrical coordinates around a plane Π (which can be interpreted as linear momentum).

To conclude, let us indicate some problems that deserve further attention. First, it would be interesting to introduce techniques to face up to actions in $n = d + 1$ dimensions ($d \geq 4$) whose Lagrangians depend linearly on the curvature and study what kind of trajectories of the relativistic particles appear in this model. Second, one might study actions involving higher order curvatures.

Acknowledgements

This work has been partially supported by the grants DGI-FEDER MTM2004-04934 and Fundación Séneca 00625/PI/04.

References

[1] R. Capovilla, C. Chryssomalakos, J. Guven, Hamiltonians for curves, J. Phys. A: Math. Gen. 35 (31) (2002) 6571–6587.

[2] G. Arreaga, R. Capovilla, J. Guven, Frenet–Serret dynamics, Classical Quantum Gravity 18 (23) (2001) 5065–5083.
[3] R. Capovilla, J. Guven, E. Rojas, Hamiltonian Frenet–Serret dynamics, Classical Quantum Gravity 19 (8) (2002) 2277–2290.
[4] M.S. Plyushchay, Massless point particle with rigidity, Modern Phys. Lett. A 4 (9) (1989) 837–847.
[5] M.S. Plyushchay, Relativistic massive particle with higher curvatures as a model for the description of bosons and fermions, Phys. Lett. B 235 (1–2) (1990) 47–51.
[6] M.S. Plyushchay, Relativistic particle with torsion, Majorana equation and fractional spin, Phys. Lett. B 262 (1) (1991) 71–78.
[7] M.S. Plyushchay, The model of the relativistic particle with torsion, Nuclear Phys. B 362 (1–2) (1991) 54–72.
[8] M.S. Plyushchay, The model of a free relativistic particle with fractional spin, Internat. J. Modern Phys. A 7 (28) (1992) 7045–7064.
[9] Y.A. Kuznetsov, M.S. Plyushchay, The model of the relativistic particle with curvature and torsion, Nuclear Phys. B 389 (1) (1993) 181–205.
[10] Y.A. Kuznetsov, M. Plyushchay, Tachyonless models of relativistic particles with curvature and torsion, Phys. Lett. B 297 (1992) 49.
[11] Y.A. Kuznetsov, M.S. Plyushchay, $(2+1)$-dimensional models of relativistic particles with curvature and torsion, J. Math. Phys. 35 (6) (1994) 2772–2784.
[12] A. Nersessian, E. Ramos, Massive spinning particles and the geometry of null curves, Phys. Lett. B 445 (1–2) (1998) 123–128.
[13] A. Nersessian, E. Ramos, A geometrical particle model for anyons, Modern Phys. Lett. A 14 (29) (1999) 2033–2037.
[14] R. Capovilla, J. Guven, E. Rojas, Null Frenet–Serret dynamics, Gen. Relativity Gravitation 38 (4) (2006) 689–698.
[15] M. Barros, A. Ferrández, P. Lucas, M.A. Meroño, General helices in the three-dimensional Lorentzian space forms, Rocky Mountain J. Math. 31 (2) (2001) 373–388.
[16] M. Barros, A. Ferrández, P. Lucas, Conformal tension in string theories and M-theory, Nuclear Phys. B 584 (3) (2000) 719–748.
[17] M. Barros, A. Ferrández, P. Lucas, M.A. Meroño, A criterion for reduction of variables in the Willmore-Chen variational problem and its applications, Trans. Amer. Math. Soc. 352 (7) (2000) 3015–3027.
[18] A. Nersessian, R. Manvelyan, H.J.W. Müller-Kirsten, Particle with torsion on 3d null-curves, Nuclear Phys. B Proc. Suppl. 88 (2000) 381–384. Constrained dynamics and quantum gravity, 1999 (Villasimius).
[19] A. Ferrández, A. Giménez, P. Lucas, Geometrical particle models on 3D null curves, Phys. Lett. B 543 (3–4) (2002) 311–317.
[20] A. Ferrández, A. Giménez, P. Lucas, Null helices in Lorentzian space forms, Internat. J. Modern Phys. A 16 (30) (2001) 4845–4863.
[21] A. Ferrández, A. Giménez, P. Lucas, Relativistic particles with rigidity along light-like curves, in: Horizons in World Physics, vol. 245, Nova Science Publishers Inc., USA, 2004, pp. 135–150.
[22] R. Conte, M. Musette, C. Verhoeven, Explicit integration of the Hénon-Heiles Hamiltonians, J. Nonlinear Math. Phys. 12 (Suppl. 1) (2005) 212–227.
[23] S. Wojciechowski, Separability of an integrable case of the Hénon–Heiles system, Phys. Lett. A 100 (6) (1984) 277–278.
[24] W.B. Bonnor, Null curves in a Minkowski spacetime, Tensor (N.S.) 20 (1969) 229–242.